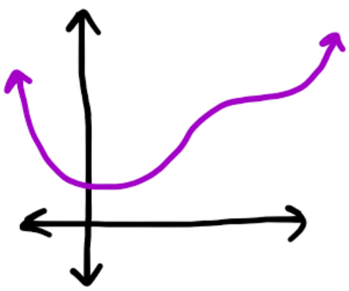
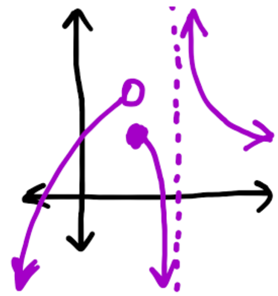
# §1.5—Continuity Continued & IVT



A continuous function



A non-continuous function

To review the **3-step definition** of continuity at a point,

> A function $f(x)$ is **continuous at a point** $x=c$ if
>
> $$\lim_{x\to c^-} f(x) = f(c) = \lim_{x\to c^+} f(x)$$

**Example 1:**

Determine the values at which the function $f(x)=\begin{cases} \dfrac{1}{x+3}, & x<-2 \\ 2x+5, & -2\le x<2 \\ x^2, & 2\le x<4 \\ 8\sqrt{x}, & x>4 \end{cases}$ is continuous, then find $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$.

**Continuity on an interval**

A function $f(x)$ is continuous on an interval *I*, if and only if the function $f(x)$ is continuous at **every** point in the interval *I*.

More specifically,

**Continuity on an open interval**

A function $f(x)$ is **<u>continuous on an open interval</u>** $(a,b)$ if and only if the function $f(x)$ is continuous at **every** point in the interval $(a,b)$.

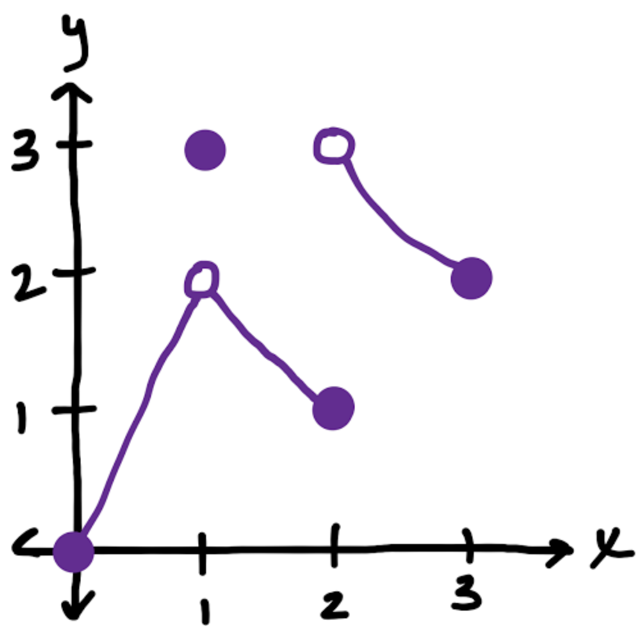
Additionally,

**Continuity on a closed interval**

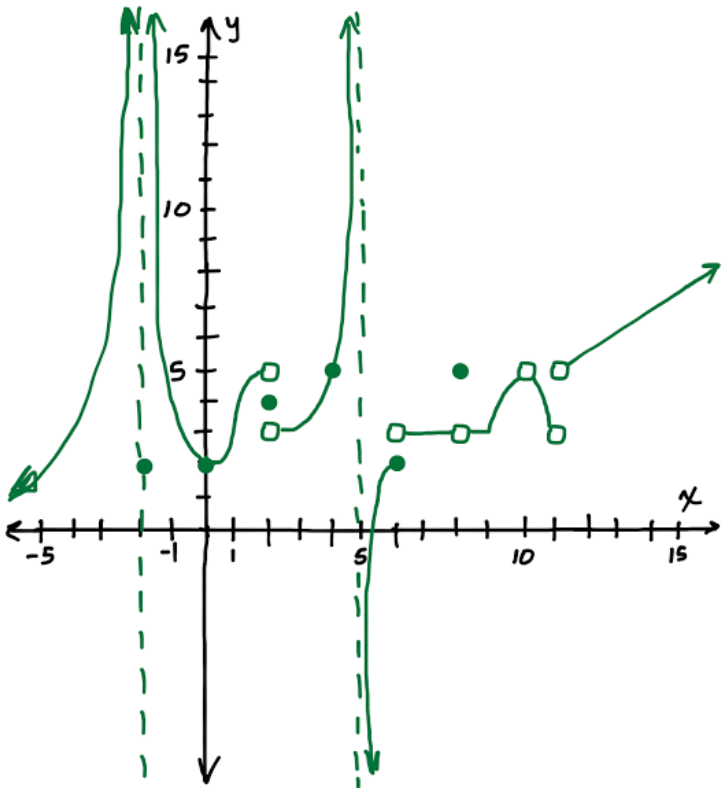
A function $f(x)$ is **<u>continuous on a closed interval</u>** $[a,b]$ if it is continuous on the open interval $(a,b)$ and if the endpoints exhibit the following:

$$\text{(i)} \lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \text{(ii)} \lim_{x\to b^-} f(x) = f(b)$$

**Example 2:**
The graph of $f(x)$ is given below. Determine the largest intervals $0 \le x \le 3$ for which the function $f(x)$ is continuous.



**Example 3:**
The graph of a familiar function $f(x)$ is given below. Answer the following questions based on the graph of $f(x)$.




(a) On the interval $x \in (-\infty, \infty)$, list the largest intervals for which $f(x)$ is continuous.

(b) Find the largest value of *b* such that the function is continuous on $(5, b]$ but not in $(5, b+1]$. Find all values of *b* that work.

(c) Give the smallest value of *a* such that the function is continuous on the largest possible interval $(a, 10)$. What other values could *a* be?

(d) Find the smallest value *k* such that the function is continuous on $(k, \infty)$.

(e) Find the smallest value *k* such that the function is continuous on $[k, \infty)$.

We saw in section 1.1 that we could easily evaluate limits of continuous functions at specific *x*-values by direct substitution. At this time, it's worth listing the families of functions that are continuous **for all values in their domain**.

**Continuous functions *over their domains***

| |
|---|
| *polynomials rational functions root functions |
| trigonometric functions (VAs are not in domain), especially *sine & *cosine |
| inverse trig functions, especially *arctangent |
| *exponential functions logarithmic functions |

(The functions with * are continuous for all real numbers)

This is why we can, for any value in their domain, evaluate limits of these functions by direct substitution. But sometimes we want to find limits of or discuss continuity of more complicated functions.

By combining continuous functions in algebraic ways, we can create new continuous functions that are also continuous over their domains.

**Example 4:**
Give the intervals for which the following functions are continuous by finding their domains.

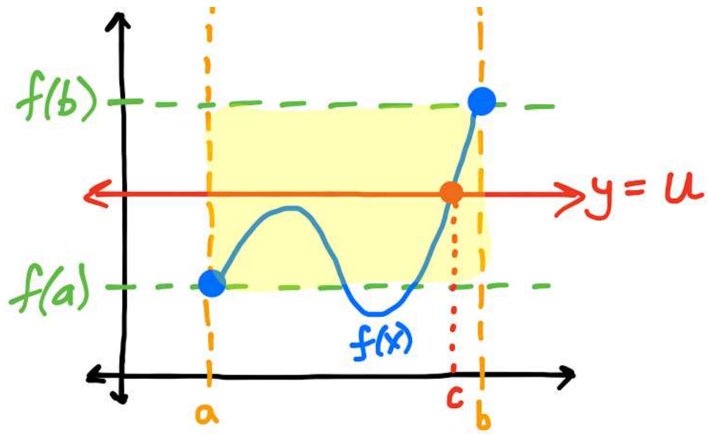
(a) $h(x) = \sin(x^2)$

(b) $h(x) = \ln(\cos(x) + 1)$

The continuity of functions plays a very important role in determining graphical and its consequences. These consequences are expressed by way of theorems. Here is an important one that relies on the continuity of a function.

**The Intermediate Value Theorem (IVT)**

**If** a function $y = f(x)$ is continuous on a closed interval $[a,b]$, **then** $f(x)$ takes on every value between $f(a)$ and $f(b)$ on that interval. Put differently, if we know of a *y*-value, say $y = u$, that resides between the *y*-values of the two endpoints, $f(a)$ and $f(b)$, then we are guaranteed at least one *x*-value, $x = c$, between the endpoints that generates that *y*-value, such that $f(c) = u$.




For a theorem to apply, the "if" part, known as the **<u>hypothesis</u>**, must be met before we can apply the "then" part, the consequence or **<u>conclusion</u>** of the theorem.

**Example 5:**
If you measure the outdoor temperature on your back porch to be $47°F$ one morning at 8:15 A.M., then measure the temperature at the same spot again 4 hours later to be $62°F$, explain why there must be a time between 8:15 A.M and 12:15 P.M where the temperature on your back porch on that day had to be $51.351°F$. Based on your two readings, is there a way to determine the exact time the temperature was $51.351°F$? Why or Why not?

Theorems like the IVT are called “existence theorems.” They prescribe the requisite conditions in which an event will happen without explicitly giving us the means to find when it will happen. To determine the latter, we would need more information about the situation.

**Example 6:**
*Piecewise Fun*: Determine if the IVT applies to the following function on the following intervals. Explain, specifically why or why not in each case.

$$f(x)=\begin{cases} -2x+1, & x\le -2 \\ x^2+1, & -2<x<1 \\ 3\sqrt{x}, & 1\le x<4 \\ \dfrac{6}{5-x}, & x\ge 4 \end{cases}$$

(a) $[-4,-2]$ (b) $[-3,0]$ (c) $[-1,1]$ (d) $[1,4.5]$ (e) $[2,7]$

We can use the IVT to prove the existence of a value, but to do this, we have to say whether the IVT applies first.

**Example 7:**
The velocity of a particle moving horizontally along the *x*-axis is given by the continuous function $v(t)$, where $t$ is measured in seconds and $v(t)$ is measured in feet per second. Selected values of $v(t)$ are given below.

| $t$ | 0 | 2 | 5 | 8 |
|---|---|---|---|---|
| $v(t)$ | –3 | –0.5 | 2 | 1 |

Explain why there must be a time on the interval $2<t<5$ where the particle is at rest.

**Example 8:**
Show that there is a positive real solution of the equation $x^5 = x + 2$

**Example 9:**
If $f(x) = x^2 + \cos(\pi x)$, prove that there is a solution for $f(x) = 4$ for some $x \in (0, 2)$.

Here's a modified part of an example from the Free Response portion of the 2007 AP exam.

**Example 10:**

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| 1 | 6 | 2 |
| 2 | 9 | 3 |
| 3 | 10 | 4 |
| 4 | –1 | 6 |

The functions *f* and *g* are continuous for all real numbers. The table above gives values of the functions at selected values of *x*. The function *h* is given by $h(x) = f(g(x)) - 6$.

Explain why there must be a value *r* for $1 < r < 3$ such that $h(r) = -5$.

Scoring Guidelines to 2007 AB3, part (a)

(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$
$h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$
Since $h(3) < -5 < h(1)$ and $h$ is continuous, by the Intermediate Value Theorem, there exists a value $r$, $1 < r < 3$, such that $h(r) = -5$.

$2 : \begin{cases} 1 : h(1) \text{ and } h(3) \\ 1 : \text{conclusion, using IVT} \end{cases}$

Scoring Statistics to 2007 AB Calculus Exam

## Free-Response Questions Scoring Statistics

| Question | Mean | Standard Deviation | Number of Points Possible |
|---|---|---|---|
| 1 | 4.33 | 2.91 | 9 |
| 2 | 3.03 | 2.24 | 9 |
| 3 | 0.96 | 1.57 | 9 |
| 4 | 2.91 | 2.73 | 9 |
| 5 | 2.48 | 2.45 | 9 |
| 6 | 3.49 | 2.42 | 9 |